Minimal Model Program

Learning Seminar

Week 8:

- Adjunction
- Inversion of adjunction
- Duality theory.

Duality Theory.

Aim: Given a projective variety X define a sheaf $\underline{\underline{\omega_X}}$ such that $H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X))^\vee$ holds for every Cohen-Macaulay sheaf $\mathcal{F}$.

dualizing sheaf.

Serre duality: $H^n(\mathbb{P}^n, \mathcal{F})$ is dual to $(\mathcal{F}$ q-coh$)$.

$\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}))$ are dual vector spaces.

If $\mathcal{F}$ is locally free, then

$$H^i(\mathbb{P}^n, \mathcal{F}) \simeq H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}) \otimes \mathcal{F}^*)^\vee.$$

Dualizing sheaf: ω_X coherent on X with a surjection

$\mathrm{Trace}_X : H^n(X, \omega_X) \longrightarrow k$ such that for every $\mathcal{F}$ coherent,

Trace_X induces a k-isomorphism:

$$\mathrm{Hom}_k(\mathcal{F}, \omega_X) \cong \mathrm{Hom}(H^n(X, \mathcal{F}), k).$$

The pair $(\omega_X, \mathrm{Trace}_X)$ is unique if it exists.

Proposition: Let $f: X \longrightarrow Y$ be a finite morphism. $\mathcal{F}$ coherent sheaf on X and $\mathcal{G}$ coherent sheaf on Y.

$f^! \mathcal{G} := \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{G})$ with the natural $\mathcal{O}_X$-module structure.

(1) There exists a $f_*\mathcal{O}_X$-isomorphism

$$f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^! \mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{F}, \mathcal{G}).$$

(2) In particular, there is a natural k-isomorphism:

$$\mathrm{Hom}_X(\mathcal{F}, f^! \mathcal{G}) = \mathrm{Hom}_Y(f_* \mathcal{F}, \mathcal{G}).$$

Proof: $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$,

$M = \Gamma(X, \mathcal{F})$, $N = \Gamma(Y, \mathcal{G})$.

$$\begin{array}{ccc} \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) & \simeq & \mathrm{Hom}_B(M, N) \\ \cup & & \cup \\ \psi & & \phi \end{array}$$

$$\psi(m): a \longmapsto \phi(am)$$

for every $a \in A$ and $m \in M$.

Proposition: Let $f : X \to Y$ (surjective) be a finite morphism of proper schemes both of pure dimension n. If ω_Y exists, then ω_X exists and $\omega_X \cong f^! \omega_Y$.

Proof: Define $\omega_X := f^! \omega_Y$.

$$
\begin{aligned}
\mathrm{Hom}_X(\mathcal{F}, \omega_X) &\simeq \mathrm{Hom}_X(\mathcal{F}, f^!\omega_Y) \\
&\simeq \mathrm{Hom}_Y(f_*\mathcal{F}, \omega_Y) \\
&\simeq H^n(Y, f_*\mathcal{F})^\vee \\
&\simeq H^n(X, \mathcal{F})^\vee.
\end{aligned}
$$

ω_Y is a right adjoint for push-forward.

ω_Y is a dualizing sheaf.

$R^i f_* \mathcal{F} = 0$ for $i > 0$.

Corollary: ω_X exists and is S_2 for every projective variety X over k.

Proof: We have $f : X \to \mathbb{P}^n_k$ surj & finite.

$$
\begin{array}{ccl}
X & \hookrightarrow & \mathbb{P}^N \\
 & & \cup \\
 & & \mathbb{P}^{N-1} \\
 & & \vdots \\
 & & \cup \\
 & & \mathbb{P}^{\dim(X)}
\end{array}
\qquad f : X \to \mathbb{P}^{\dim(X)}
$$

$\omega_{\mathbb{P}^n_k}$ exists, hence ω_X exists.

ω_X is S_2 $\iff$ $f_*\omega_X$ is S_2.

$f_*\omega_X$ is $\mathcal{H}om$ to a locally free shf. so it is S_2. $\square$

Corollary: X proj scheme of pure dim n over k. $\mathcal{F}$ is a coherent sheaf on X such that $\operatorname{supp}\mathcal{F}$ has pure dimension n. Then the following conditions are satisfied:

(1) If $\mathcal{F}$ is CM, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ is CM

The converse is true if $\mathcal{F}$ is a S_2 sheaf.

(2) If X is S_2, then $\mathcal{O}_X$ is CM iff ω_X is CM

Proof: 1) $f: X \longrightarrow \mathbb{P}^n$ surjective and finite

$\mathcal{F}$ is S_2.

$\mathcal{F}$ is CM $\iff$ $f_*\mathcal{F}$ is locally free

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ is CM $\iff$ $f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$

is

$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \omega_{\mathbb{P}^n})$

$+ S_2$.

Duality Theory:

Theorem: X projective scheme of pure dim n over a field k. $\mathcal{F}$ a CM sheaf on X such that $\operatorname{supp}\mathcal{F}$ has pure dimension n.

Then, we have an isomorphism:

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X))^\vee.$$

Proof: $f: X \longrightarrow \mathbb{P}^n = P$ be a finite surj morphism

$$H^i(X, \mathcal{F}) \simeq H^i(P, f_*\mathcal{F}),$$

$f_*\mathcal{F}$ is locally free since $\mathcal{F}$ is CM, + $\operatorname{supp}\mathcal{F} = X$.

serre dual for locally free

$$H^i(P, f_*\mathcal{F}) \simeq H^{n-i}(P, \mathcal{H}om_{\mathcal{O}_P}(f_*\mathcal{F}, \omega_P))^\vee$$

by Serre duality, and furthermore

$$H^{n-i}(P, \mathcal{H}om_{\mathcal{O}_P}(f_*\mathcal{F}, \omega_P)) \simeq$$

right adj.

$$H^{n-i}(P, f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)) \simeq$$

ω_X is a dualizing sht.

$$H^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)).$$

$H^i(X, \mathcal{F})$ is dual to $H^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X))$.

Proposition: X projective CM scheme of pure dim n over a field k and $D \subseteq X$ Cartier divisor. Then

$$\omega_D \simeq \omega_X(D) \otimes \mathcal{O}_D.$$

Proposition: Let X be a normal projective variety of dim n over a field k. Then $\omega_X \simeq \mathcal{O}_X(K_X)$.

Adjunction formula: X smooth & $D \subseteq X$ smooth Cartier. Then $(K_X + D)|_D = K_D$.

Aim: Compare singularities of $(X, B+S)$ log pair with those of $(S, B|_S)$.

Weil. reduced.

Remark: If S is Cartier in codimension two, then

$$(K_X + B + S)|_S = K_S + B|_S.$$

If you have quotient singularities in codimension two.

Shokurov 3-fold flips S.2

$$(K_X + B + S)|_S = K_S + B|_S + \sum_{P_i \in S} \left(1 - \frac{1}{m_i}\right) P_i$$

$$\begin{array}{c} \mathbb{C}^2 \curvearrowleft \mu_{m_i} \\ \big\downarrow {\scriptstyle /\mathbb{Z}/m_i\mathbb{Z}} \\ X \end{array}$$

where P_i is a cod two point of X which is orbifold of order m_i.

Adjunction: If the sing of $(X, B+S)$ are nice then the sing of $(S, B|_S)$ are nice.

Inv of adjunction: If $(S, B|_S)$ has nice sing then so does $(X, B+S)$.

Proposition: X normal, S normal Weil carbier in cod 2.

$- \subseteq S$ closed $B = \sum' b_i B_i \geq 0$ $\mathbb{Q}$-divisor

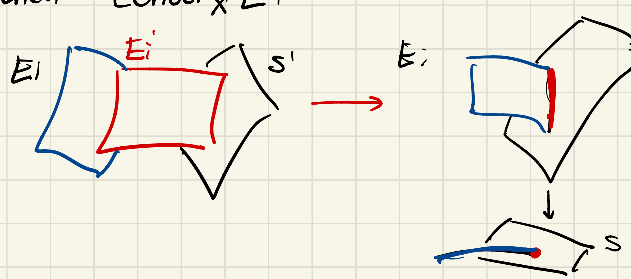
Assume $K_X + S + B$ is $\mathbb{Q}$-Carbier. Then:

$$\text{totaldiscrep}(\text{center} \subseteq Z, S, B|_S) \geq$$
$$\text{discrep}(\text{center} \subseteq Z, X, S+B) \geq$$
$$\text{discrep}(\text{center} \cap S \subseteq Z, X, S+B)$$

$(X, B+S)$ plt
$\Downarrow$
$(S, B|_S)$ klt

Proof: $f: Y \longrightarrow X$ log resolution of $(X, S+B)$

$S' = f_*^{-1} S$, $f_*^{-1}(S+B)$ is smooth, E_i exc which intersects S' then $\text{center}_X E_i \subseteq S$.



$$K_Y + S' \equiv f^*(K_X + S + B) + {\textstyle\sum_i}' e_i E_i .$$

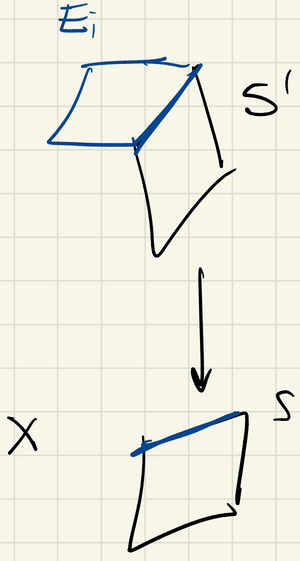
adjunction

$$K_{S'} = K_Y + S'|_{S'} \qquad K_X + S + B|_S = K_S + B|_S$$

$$K_{S'} \equiv f^*(K_S + B|_S) + {\textstyle\sum_i}' e_i (E_i \cap S').$$

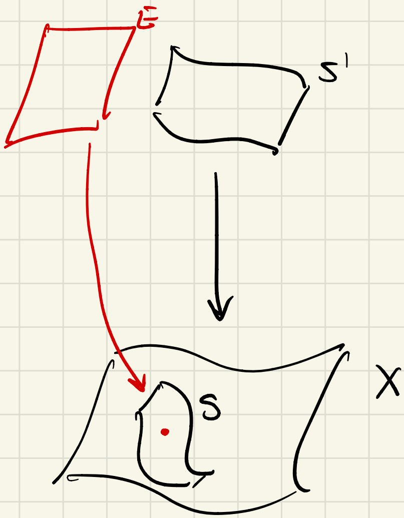
S' is disjoint from $f_*^{-1}B$, hence if $E_i \cap S' \neq \emptyset$, then E_i is f-exc and $\mathrm{center}_X E_i \subseteq S$.

Hence, we showed that every log discr. which happens in $S' \longrightarrow S$ also happens in $Y \longrightarrow X$ in such a way that the center lies inside S. $\square$



If E_i is exc over X, then $E_i \cap S'$ may not be exc over S.

Problem: discrep of $(S, B|_S)$ are nice but those of $(X, B+S)$ are bad. E exceptional over X with $a_E(X, B+S) < 0$.



Y log resolution of X.

$E \cap S' = \emptyset$

Philosophy: In a Fano object, the locus which is most singular is connected.

The theorem that realizes principle is called Kollár–Shokurov connectedness Theorem.

Definition: (X,B) a log pair.

$Z \subseteq X$ is a log canonical center if there exists E over X with

- $a_E(X,B)=0$
- $c_E(X)=Z$.

We denote by $LCC(X,B)$ the union of all lcc's.
It is a closed subset.

Theorem (Kollár–Shokurov connectedness):

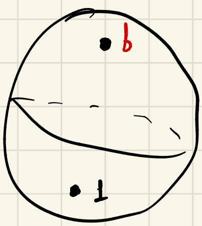
$X \to Z$ proj morphism, (X,B) lc, $-(K_X+B)$ ample over Z, then $LCC(X,B)$ is connected over Z.

$\mathbb{Q}$-trivial

big + nef suffices.

at most two comp

Example:



$\mathbb{P}^1$ $\deg K_{\mathbb{P}^1} = -2$.

$b_1+b_2+b_3<2$

$b \longrightarrow 1$. $b=1$.

Remarks: field k is C_1 is every homogeneous $f(x_0,\dots,x_n) \in k[x_0,\dots,x_n]$ of $\deg \leq n$ has a non-trivial zero. (Fano variety)

PAC: every geometrically integral k-var has a k-point.

Ax: Is every PAC field of char 0 C_1? Yes.

Theorem: Let $f: Y \to X$ be a proper birational morphism. Y smooth, X normal. Let $D = \sum_i d_i D_i$ sne $\mathbb{Q}$-divisor on Y such that $f_* D \geq 0$ and $-(K_Y + D)$ is f-nef. Write

$$A = \sum_{i : d_i < 1} d_i D_i \quad \& \quad F = \sum_{i : d_i \geq 1} d_i D_i .$$

Then $\operatorname{Supp} F = \operatorname{supp} \lfloor F \rfloor$ is connected in a neighborhood of any fiber of f.

Proof: $\lceil A \rceil - \lfloor F \rfloor = K_Y - f^*(K_X + D) + \{A\} + \{F\}$.

By KV vanishing $R^1 f_* \mathcal{O}_Y(\lceil A \rceil - \lfloor F \rfloor) = 0$

$$0 \to \mathcal{O}_Y(\lceil A \rceil - \lfloor F \rfloor) \to \mathcal{O}_Y(\lceil A \rceil) \to \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \to 0$$

connected.

$$\underset{\mathcal{O}_X}{\underset{\|}{f_* \mathcal{O}_Y(\lceil A \rceil)}} \twoheadrightarrow f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil).$$

will have ≥ 2 non-trivial summands

$\lceil A \rceil$ is f-exceptional, eff $\Longrightarrow f_* \mathcal{O}_Y(\lceil -A \rceil) = \mathcal{O}_X$.

$f_* D \geq 0$

$\lfloor F \rfloor = F_1 \cup F_2$ in a neighborhood of $f^{-1}(x)$.

$$f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_{(x)} \simeq f_* \mathcal{O}_{F_1}(\lceil -A \rceil)_{(x)} + f_* \mathcal{O}_{F_2}(\lceil A \rceil)_{(x)}$$

$$\mathcal{O}_{X,x} \twoheadrightarrow f_* \mathcal{O}_{F_1}(\lceil -A \rceil)_{(x)}$$

$\square$

Theorem: Let X be a normal variety and $S \subseteq X$ normal Weil divisor which is Cartier in cod 2. $B \geq 0$ $\mathbb{Q}$-divisor such that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then

(1) $(X, S+B)$ is plt near S iff $(S, B|_S)$ is klt

(2) Assume in addition B is $\mathbb{Q}$-Cartier and S is klt.

Then $(X, S+B)$ is lc near S iff $(S, B|_S)$ is lc

Proof: $\Longrightarrow)$ is a previous prop.

We need to prove $\Longleftarrow)$ $\quad g: Y \longrightarrow X$ log resolution of $(X, S+B)$.

$K_Y + D \equiv g^*(K_X + S + B)$, $\quad S' = g_*^{-1} S$, $\quad F = S' + F'$.

where F' contains a comp of coeff ≥ 1.

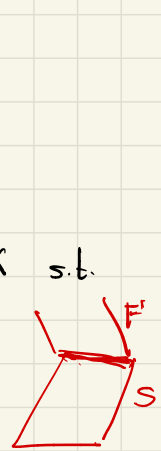
By adjunction $K_{S'} = g^*(K_S + B|_S) + (A - F')|_S$.

$(X, B+S)$ is plt near S iff $F' \cap g^{-1}(S) = \emptyset$.

$(S, B|_S)$ is klt iff $(F')^{\geq 1} \cap S' = \emptyset$.

By K_S connectedness, $x \in S$, there exists $U_x \subseteq X$ s.t.

$(S' \cup (F')^{\geq 1}) \cap g^{-1}(U_x)$ is connected,



hence $F' \cap g^{-1}(U_x) = \emptyset$, moving x around S

we finish the proof of the statement $\square$.

Adjunction to higher codimension centers:

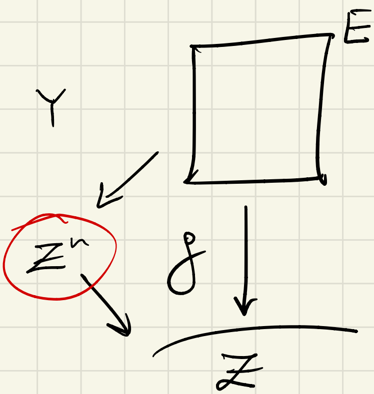
$(X, S+B)$ sing $\Longleftrightarrow$ $(S, B|_S)$ sing

(X,B), Z is a log canonical center of (X,B).

$\operatorname{codim}(Z,X) = 1$.

$\operatorname{codim}(Z,X) \geq 2$.

define (Z, B_Z) such that the sing of (X,B) around Z can be compared to those of (Z, B_Z)?



$$g^*(K_X+B)|_E = K_Y + B_Y + E|_E$$

$$\| $$

$$K_E + B_E.$$

is trivial over Z.

$$K_E + B_E \sim_{\mathbb{Q},Z} 0.$$

we want to write

$$K_E + B_E = g^*(K_Z + B_Z + M_Z).$$

measures the sing of the fibration

measures the variation of moduli.